B3 - Stochastic Processes : End-Semester Exam

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May 2, 2025. Time : 10.00 AM - 12.30 PM. **Maximum points : 40**

2 points will be deducted if you do not write your name on the answerscript.

There are two parts to the question paper - PART A and PART B. Read the instructions for each part carefully. Some simple notions and notations are recalled at the end.

PART A : MULTIPLE-CHOICE QUESTIONS - 10 Points.

Please write only the CORRECT CHOICES in your answer scripts. No explanations are needed. Write PART A answers in a separate page.

1. Let X be a non-negative random variable with mean and variance 1. Which of the following statements are always true ?

   A. $\mathbb{P}(X > 0) \geq \frac{1}{4}$

   B. $\mathbb{P}(X \geq 46) \leq \frac{1}{2025}$

   C. $\mathbb{P}(X = 0) \leq \frac{1}{2}$

   D. $\mathbb{P}(X \geq 3) \geq \frac{1}{3}$

2. Suppose a discrete-time Markov chain has the transition matrix

$$P = \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0 \end{bmatrix}.$$

   The number of communication classes and period of the chain are respectively -

   A. (1,1)

   B. (2,1)

   C. (1,2)

   D. (2,2)

3. A Galton-Watson branching process starts with one individual. Each individual produces 0, 1, 2 or 3 offspring with equal probabilities and independent of other individuals. What is the expected number of individuals in the second generation?

   A. 3/2
   B. 3
   C. 6
   D. 9/4

4. Let $X_n, n \geq 0$ be a martingale sequence. Which of the following are correct ?

   A. $\mathbb{E}[X_n] - X_n, n \geq 0$ is a martingale sequence.
   B. $X_n^2 - n, n \geq 0$ is a martingale sequence.
   C. $1 - X_n^2, n \geq 0$ is a sub-martingale sequence.
   D. $X_n^2 + \mathbb{E}[X_n^2], n \geq 0$ is a sub-martingale sequence..

5. Consider the Markov chain with transition matrix

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

   Which of the following statements are correct?

   A. The Markov chain is irreducible.
   B. The stationary distribution is unique.
   C. Every stationary distribution π satisfies $\pi(1) = \pi(2)$.
   D. The Markov chain is reversible with respect to some probability distribution π.

PART B: SHORT ANSWERS - 30 Points.

ALL QUESTIONS CARRY 10 POINTS. ATTEMPT ANY THREE OF THEM ONLY.

You are free to use any results that you have learnt in your probability courses but please cite them clearly. Provide as many details as you can.

1. Let $\mathbf{X} = \mathbf{X}_1 \times \ldots \mathbf{X}_n$ and $f, g : \mathbf{X} \to \mathbb{R}$ be functions. Let $X_1, \ldots, X_n$ be independent random variables taking values in $\mathbf{X}_1, \ldots, \mathbf{X}_n$ respectively. For $A \subset [n]$, let X^A denote the vector

$$X_i^A := X_i' \mathbf{1}[i \in A] + X_i \mathbf{1}[i \notin A],$$

where X_i''s are independent copies of X_i's. In other words, we replace X_i's in A by independent copies. Further define

$$\Delta_i f(X^A) = f(X^{A \cup i}) - f(X^A), A \subset [n], i \notin A.$$

For $i \in A$, set $\Delta_i f(X^A) = 0$ trivially. Show that

$$\mathsf{Cov}\,(f(X), g(X)) = \frac{1}{2n} \sum_{j=1}^{n} \sum_{A \subset [n]\,, j \notin A} \frac{1}{\binom{n-1}{|A|}} \mathbb{E}\big[\Delta_i g(X) \Delta_i f(X^A)\big]\,.$$

2. Let T_d be the infinite d-ary tree rooted at o with $d \geq 2$. Edges in T_d are deleted with probability $1-p$ and retained with probability p independently of other edges. Call the subsequent graph with retained edges $T_d(p)$. Show the following:

$$\mathbb{P}\left\{\text{there exists an infinite path from } o \text{ in } T_d(p)\ \right\} = \begin{cases} 0 & \text{if } pd \leq 1 \\ > 0 & \text{if } pd > 1 \end{cases}$$

3. Let $T_d, T_d(p)$ be as above with $p \in (0,1)$ and $V_n := \{v \in T_d : d(v,o) \leq n\}$, be the set of vertices in T_d that are at most at distance n from o. Let $I(n,p)$ be the number of vertices in V_n that have no neighbours in $T_d(p)$. Is there a deterministic sequence $a_n, n \geq 1$ of non-negative numbers (possibly depending on d, p) such that

$$\lim_{n \to \infty} a_n^{-1} I(n,p) = 1, \text{ a.s.?}$$

4. Let $\lambda \in (0, \infty)$. The network $T_d(\lambda)$ is the graph T_d with the following weight function: $c(x,y) = \lambda^j$ if $d(0,y) = d(0,x) + 1 = j$ i.e., edge-weights are λ^j on edges between vertices at level $j-1$ to j. Show that the Random walk on $T_d(\lambda)$ is transient iff $d\lambda > 1$.

5. Let (G,c) be a finite network on vertex set V and $a \in V, Z \subset V$ such that $\{a\} \cup Z \subsetneq V$. Let i be the unit current flow from a to Z and v be the corresponding voltage function. Show that

$$c(x)v(x) = \mathcal{G}_{\tau_Z}(a,x),\ \forall x \in V$$

where $\mathcal{G}_{\tau_Z}$ is *stopped Green's function* defined as

$$\mathcal{G}_{\tau_Z}(a,x) := \mathbb{E}_a\Big[\sum_{t=0}^{\tau_Z - 1} \mathbf{1}[X_t = x]\Big],$$

with τ_Z being the hitting time of Z.

Notions and Notation-

T_d, d-ary Tree: For $d \geq 2$, let T_d be an infinite d-ary tree rooted at o i.e., o has d-neighbours, each of whom have d-further neighbours and so on. Level j vertices consist of those vertices at distance j from the root.

Hölder's inequality: Let $w_j \geq 0$, $\sum_{j=1}^n w_j \leq 1$ be weights and Y_j's be non-negative random variables such that $\mathbb{E}\left[Y_j^{1/w_j}\right] < \infty$ for all j. Then, we have that

$$\mathbb{E}\left[\prod_{j=1}^n Y_j\right] \leq \prod_{j=1}^n \mathbb{E}\left[Y_j^{1/w_j}\right]^{w_j}.$$

Stirling's Approximation: $\sqrt{2\pi n}\,\left(\frac{n}{e}\right)^n e^{1/(12n+1)} \leq n! \leq \sqrt{2\pi n}\,\left(\frac{n}{e}\right)^n e^{1/(12n)}$.

Hitting time: For a Markov chain X_t on state space V, and $W \subset V$, we define hitting time of W as $\tau_W := \inf\{t \geq 0 : X_t \in W\}$.

Harmonic functions: For a bounded $f : V \to \mathbb{R}$ and stochastic matrix P, define the Laplacian operator as $\Delta f(x) := \sum_y P(x,y)f(y) - f(x)$. A function f is *harmonic on* $W \subset V$ if $\Delta f(x) = 0$ for all $x \in W$.